Shock Spectra for Nonlinear Spring-Mass Systems and Their Applications to Design

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A method is discussed for presenting response spectra for a nonlinear system, and applications are indicated for analysis and design problems. Typical results are presented for nonlinear systems having cubic-softening and bilinear spring characteristics.

Nomenclature

$A(\omega),D(\omega),$ $V(\omega)$	= absolute acceleration spectrum, relative displacement spectrum, and pseudovelocity spectrum of a linear system
$f(t)$	= $\ddot{s}(t)/\|\ddot{s}\|_m$ = functional shape of the ground acceleration pulse
K	= spring constant of associated linear system
K_1,K_3	= spring constants used in the nonlinear terms of the bilinear and the cubic-softening springs, respectively
m	= mass that is supported on the spring of the system
$P(Y)$	= spring force of the nonlinear system
$R(Y)$	= nonlinear portion of $P(Y)$, see Eq. (12)
$\ddot{s}(t)$	= $\alpha f(t)$ = ground acceleration pulse
$S(\omega)$	= amplification spectrum of the linear system
$\bar{S}(\omega,\alpha,\mu)$, $S^*(\omega,\alpha,\mu)$, $\hat{S}(\omega,\alpha,\mu)$	= nondimensional spectra of the nonlinear system, defined by Eqs. (19–21)
t_0	= duration of ground acceleration pulse
t_m	= rise time of the ground acceleration pulse
T_c	= arbitrary characteristic time
Y	= relative displacement
Y_1	= elastic limit displacement of a bilinear spring
$\ddot{X}$	= absolute acceleration of the mass
Z	= $\omega^2 Y/\alpha$ = dimensionless displacement
α	= $\|\ddot{s}(t)\|_m$ = amplitude of the ground acceleration pulse
Δ	= $\|Y\|_m$ = a specific value of the maximum relative displacement
ω	= $(k/m)^{1/2}$ = circular frequency
τ	= ωt = dimensionless time
β,λ	= nonlinearity parameters, defined by Eqs. (24) and (37), respectively
μ	= general symbol to represent any nonlinearity parameter

Subscripts

$(\ )_0$	= quantity associated with a linear system
$(\ )_m$	= the maximum value of a quantity

Introduction

SHOCK response problems arise in the aerospace industry from many causes, such as the isolation and protection of instruments or guidance and control packages, the protection of equipment during shipping and erection, the response of a vehicle in flight or on ground due to enemy attack, and the protection of a rocket in a silo against earthquake or nuclear blasts. On some occasions, shock and vibration isolation is achieved by the insertion of a set of elastic springs; on other occasions it may be advantageous to use nonlinear springs. For example, against intense shock one may wish to take advantage of the energy dissipation mechanism of crushable materials or plastic yielding of metals and to consider nonlinear "softening" springs. On the other hand, to isolate from small steady vibratory loads, one may wish to soften the support to reduce the fundamental frequency; in order to retain a reasonable sturdiness of the structure against larger loads, one may wish to stiffen the springs, thus leading to the consideration of nonlinear "hardening" springs. Or one may wish to use a spring that is hardening under moderate loads but yielding to large loads for protection against strong shocks.

To examine the shock response of a nonlinear system, in this paper a mass on a nonlinear spring is considered first. Similar treatment of multi-degree-of-freedom systems is possible but will be reported in a separate article.

For a linear single-degree-of-freedom oscillator, the representation of the peak magnitudes of various response quantities by means of shock spectra is a well-established technique (see, e.g., Refs. 1–6). On the other hand, for nonlinear systems, the use of shock spectra is not well established, although, in a few special problems, spectra of various forms have been employed.[6–9]

It is the purpose of this report to discuss several possible schemes of presenting response spectra for a simple nonlinear system and also to show how such spectra can be used in analysis and design problems. For conciseness, the discussion herein is restricted to an undamped single-degree-of-freedom system composed of a concentrated mass on a nonlinear spring. The excitation considered is a transient displacement of the base of the spring, that is, a "ground shock." With minor modification the discussion applies also to the response of the system due to a force pulse acting on the mass.

This report does not deal with the methods (which may be analytical, numerical, or experimental) that are used to determine the response; it is assumed that the pertinent response data have been obtained. This discussion of the problem is concerned with methods of presenting the response in spectrum form so as to show the effect of the system's parameters upon the dynamic behavior of the system and to show how this behavior differs from that of a linear system. How the spectra can be used in practical problems of analysis and design such as selecting a nonlinear spring to limit the peak displacement and acceleration to specified magnitudes is considered also.

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1. Response Spectra for a Linear System

Before discussing the nonlinear system, the definitions of the response spectra for an undamped linear single-degree-of-freedom system that is subjected to a ground shock (Fig. 1) will be reviewed. A response quantity of interest is the relative displacement, which is governed by the equation

$$Y + \omega^2 Y = -\ddot{s}(t) \tag{1}$$

Another response quantity of importance is the "absolute" acceleration, $\ddot{X}$. By definition, $\ddot{X} = \ddot{Y} + \ddot{s}(t)$, and by Eq. (1) this gives

$$\ddot{X}(t) = -\omega^2 Y(t) \tag{2}$$

It is convenient to express the ground acceleration $\ddot{s}(t)$ in the form

$$\ddot{s}(t) = \alpha f(t) \tag{3}$$

where

$$\alpha = \max|\ddot{s}(t)| \tag{4}$$

The *relative displacement spectrum*, $D(\omega)$, is defined to be

$$D(\omega) = \max|Y(t)| \tag{5}$$

and the *absolute acceleration spectrum*, $A(\omega)$, is defined to be

$$A(\omega) = \max|\ddot{X}(t)| \tag{6}$$

By virtue of Eq. (2), one has

$$A(\omega) = \omega^2 D(\omega) \tag{7}$$

It is also common to define a *pseudo-velocity spectrum*, $V(\omega)$:

$$V(\omega) = \omega D(\omega) = (1/\omega)A(\omega) \tag{8}$$

For an interpretation of the physical significance of $V(\omega)$, see Ref. 1, p. 10.

Since $A(\omega), V(\omega)$, and $D(\omega)$ are simply related, it is obvious that any one is sufficient to determine the other two. In fact, all three spectra can be represented by a single curve on a tricoordinate logarithmic graph (see Ref. 1, pp. 41, 73). Such a representation is not possible for a nonlinear system.

The forementioned spectra are in dimensional form; they depend upon both the functional shape of the ground acceleration and its amplitude. For compactness of presentation and also to emphasize the significant characteristics of the response which are due to the given type of shock pulse independent of its amplitude, it is often desirable to transform the spectra to dimensionless form. One particularly useful representation of this type is the so-called *amplification spectrum*, $S(\omega)$, which is the ratio of the maximum acceleration response of the mass to the maximum acceleration of the ground shock, that is,

$$S(\omega) = |\ddot{X}|_m/\alpha \tag{9}$$

By virtue of Eq. (2), the amplification spectrum also can be expressed as

$$S(\omega) = \omega^2|Y|_m/\alpha \tag{10}$$

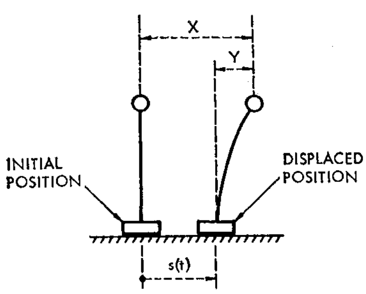


Fig. 1 Coordinates of a simple oscillator

The amplification spectra for many different types of ground acceleration pulses are available in the literature; see, e.g., Refs. 2, 4, 6, and 12.

2. Nonlinear System

For a mass on a nonlinear spring, the equation of motion is

$$m\ddot{Y} + P(Y) = -m\ddot{s}(t) \tag{11}$$

where $P(Y)$ is the spring force corresponding to a relative displacement Y. In the subsequent discussion, it is generally expedient to express $P(Y)$ as the sum of a linear term and a nonlinear term, that is, to put

$$P(Y) = K[Y + R(Y)] \tag{12}$$

The constant K is arbitrary. However, it is generally desirable to choose K equal to the slope of the $P(Y)$ curve at $Y = 0$.

On introducing (12) into (11), the governing equation becomes

$$\ddot{Y} + \omega^2[Y + R(Y)] = -\alpha f(t) \tag{13}$$

where $\omega^2 = K/m$.

In discussing the behavior of the nonlinear system, it is generally useful to relate its response to that of an associated linear system. For this linear system, choose one that has the same mass, m, as the nonlinear system and a linear spring constant, K, that is equal to the parameter K in Eq. (2). The equation of motion for this associated system is therefore

$$\ddot{Y}_0 + \omega^2 Y_0 = -\alpha_0 f(t) \tag{14}$$

Here a subscript (0) is introduced in order to distinguish the variables in the linear system from those in the nonlinear system.

For the linear system, the peak absolute acceleration is related to the peak relative displacement by Eq. (2), i.e., one has

$$|\ddot{X}_0|_m = \omega^2|Y_0|_m \tag{15}$$

The corresponding relation for the nonlinear system is somewhat more complicated. Since $\ddot{X} = \ddot{Y} + \alpha f(t)$, one finds, by virtue of Eq. (13),

$$|\ddot{X}|_m = \max|\omega^2[Y + R(Y)]| \tag{16}$$

From this, it is observed that, unlike the linear system, the maximum absolute acceleration and the maximum relative displacement may be attained at different times. However, if the spring force $P(Y)$ increases monotonically within the range of the response, these two response quantities will reach their respective peak values at the same time, and Eq. (16) takes on the simpler form

$$|\ddot{X}|_m = \omega^2[|Y|_m + R(|Y_m|)] \tag{17}$$

Now consider a problem for which the ground shock function $f(t)$, the amplitude α, and the spring force nonlinear term $R(Y)$ are given. The response magnitudes, $|Y|_m$ and $|\ddot{X}|_m$, are then functions of ω alone. In this case, a spectrum curve can be plotted for each, and these will be analogous to the $D(\omega)$ and $A(\omega)$ spectra of a linear system. Thus, for a ground shock of given functional form and amplitude and a specific nonlinear spring, there is no difficulty in presenting the data.

However, specific spectra for a situation of the forementioned type are limited in usefulness. They do not provide any information regarding the variations in response due to changes in the amplitude, α, of the ground shock. They do not provide the information needed to solve spring design problems that involve selecting the nonlinearity parameters to limit the displacement and acceleration response to specified peak values.

It is possible, of course, to plot families of curves for a range of values of the ground shock amplitude α and of all the nonlinearity parameters. The difficulties in this procedure are the large number of graphs that are required and the possibility that the families of spectra in this form will not clearly show the effects of the nonlinearities in a way that is most helpful to a designer.

Because of the forementioned difficulties, one of the objectives of this report is to investigate methods of presenting the response data more compactly and more meaningfully. The procedure adopted is to select certain dimensionless ratios and parameters that simplify the presentation and that enable the nonlinear spectra to be related closely to the amplifications spectrum of the associated linear system.

3. Nondimensional Spectra for the Nonlinear System

The nondimensional amplification spectrum, $S(\omega)$, for a linear system is defined by Eqs. (7) and (8). In terms of the notation used in Sec. 2 to describe the associated linear system, this amplification spectrum is

$$S(\omega) = |\ddot{X}_0|_m/\alpha_0 = \omega|^2 Y_0|_m/\alpha_0 \tag{18}$$

For the nonlinear system, one may define two analogous nondimensional spectra, namely, a nondimensional absolute acceleration spectrum

$$\bar{S}(\omega,\alpha,\mu) = |\ddot{X}|_m/\alpha \tag{19}$$

and a nondimensional relative displacement spectrum

$$S^*(\omega,\alpha,\mu) = \omega^2|Y|_m/\alpha \tag{20}$$

Since $|X|_m \neq \omega^2|Y|_m$, it is necessary to have these two different spectra for the nonlinear system in order to provide the same information that is contained in the single spectrum, $S(\omega)$, of the linear system.

For a given pulse shape, $f(t)$, the response spectra $\bar{S}$ and S^* are, in general, functions of the frequency ω, the ground acceleration amplitude α, and the parameters that are needed to describe the nonlinearity of the spring. In Eqs. (19) and (20) the symbol μ is used to denote the nonlinearity parameters. More than one parameter may be needed to describe a specific spring.

In some problems it is desirable to use an alternate form of the nondimensional relative displacement spectrum:

$$\hat{S}(\omega,\alpha,\mu) = |Y|_m/\alpha T_c^2 \tag{21}$$

where T_c is any convenient characteristic time. For example, one may choose T_c equal to the rise time of the ground acceleration pulse or equal to the pulse duration.

For a particular system with a given nonlinear spring and a given value of α, the $\bar{S}$ and $\hat{S}$ spectra become functions of ω alone and are the nonlinear counterparts of the linear spectra $A(\omega)$ and $D(\omega)$, respectively. However, in general, the three nonlinear spectra just defined are functions not only of ω but also of α and the nonlinearity parameters, μ. In many problems the dependence on α can be eliminated or simplified by a suitable choice of the nonlinearity parameters. In fact, it is this possibility of reducing the dependence of the nondimensional spectra upon the various parameters which makes the nondimensional forms useful.

4. Cubic Softening Spring: Parameter β

As an illustration of the relationships just derived, a system with a cubic-softening spring for which

$$P(Y) = KY - K_3Y^3 \tag{22}$$

will be considered. The corresponding equation of motion is

$$\ddot{Y} + \omega^2[Y - (K_3/K)Y^3] = -\alpha f(t) \tag{23}$$

The nonlinearity can be expressed in many different ways.

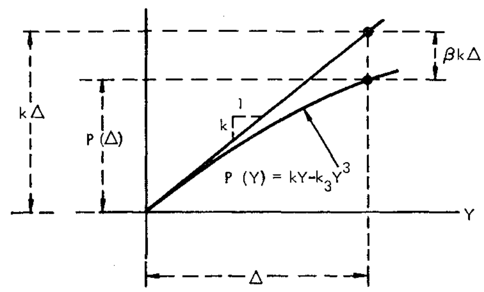


Fig. 2 Cubic-softening spring and corresponding parameter β

One way would be to take the ratio K_3/K as the parameter. The disadvantage of this scheme is that the response ratios and the nondimensional spectra are then functions of α as well as of K_3/K, and that, consequently, a great number of spectrum curves are required to represent the response of the system for a range of the parameters.

For this reason other ways are sought to specify the nonlinearity in order to simplify the presentation. One such scheme is the β-parameter representation that has been proposed by Barton and Fung.[10,11] The parameter β is defined in the following manner. Let $|Y|_m$ be denoted by Δ. Then the corresponding nonlinear spring force is $P(\Delta)$ (see Fig. 2). For the same displacement, the spring force of the associated linear system is $K\Delta$. The parameter β is defined in terms of these spring forces as

$$\beta = 1 - [P(\Delta)/K\Delta] = -[R(\Delta)/\Delta] \tag{24}$$

For a cubic-softening spring one has $R(\Delta) = -(K_3/K)\Delta^3$ and hence

$$\beta = (K_3/K)\Delta^2 \tag{25}$$

Observe that, for a given spring with a specified ratio K_3/K, the parameter β will vary depending upon the magnitude of Δ. Inversely, for a constant value of β and varying values of Δ, one will have varying values of K_3/K (see Fig. 3). Also note that, for a cubic-hardening spring, the parameter β will be negative if one uses the definition (24).

Now return to a consideration of the equation of motion (23) and rewrite it in the following form:

$$\frac{1}{\omega^2}\frac{d^2}{dt^2}\left(\frac{Y\omega^2}{\alpha}\right) + \frac{Y\omega^2}{\alpha} - \frac{\alpha^2K_3}{\omega^2K}\left(\frac{Y\omega^2}{\alpha}\right)^3 = -f(t) \tag{26}$$

Introducing the dimensionless quantities

$$Z = Y\omega^2/\alpha \qquad \tau = \omega t$$

Eq. (26) becomes

$$(d^2Z/d\tau^2) + Z - CZ^3 = -f(\tau/\omega) \tag{27}$$

where

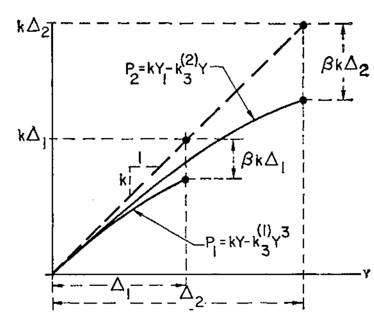
$$C = \beta/(|Z|_m)^2 \tag{28}$$

Note that

$$|Z|_m = \omega^2|Y|_m/\alpha = S^* \tag{29}$$

For specific values of ω and C, one may solve (27) to obtain

Fig. 3 Cubic-softening springs with same β and different Δ's



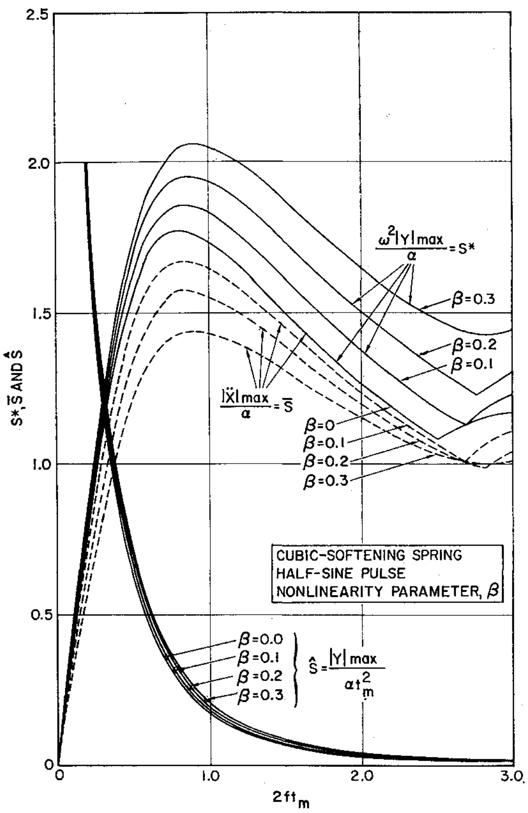


Fig. 4 Shock spectra; cubic-softening spring

$|Z|_m$. Substituting this value of $|Z|_m$ into Eq. (28), one finds the corresponding value of β to be

$$\beta = (|Z|_m)^2 \cdot C \tag{30}$$

Repeating this calculation for a range of values of the parameters ω and C, one can obtain sufficient data to plot S^* (which is the same as $|Z|_m$) as a family of curves with ω as the abscissa and β the family parameter.

Observe that the determination of S^* is independent of α, and hence this shock spectrum is a function only of ω and β, that is, $S^* = S^*(\omega,\beta)$.

The absolute acceleration spectrum $\bar{S}$ could be obtained by monitoring the solution of Eq. (27) to locate the maximum of $|Z - CZ^3|$, that is,

$$\begin{aligned}\bar{S}(\omega,\beta) &= \max|Z - CZ^3| \\ &= \max|Z - [\beta Z^3/(|Z|_m)^2]|\end{aligned} \tag{31}$$

In most practical applications, the maximum displacement will be restricted to the monotonic range of the spring force (i.e., $\beta \leq \frac{1}{3}$). In such cases, Eq. (31) reduces to the simple expression

$$\bar{S}(\omega,\beta) = (1 - \beta)\cdot S^*(\omega,\beta) \tag{32}$$

and one can determine $\bar{S}$ merely by multiplying S^* by the quantity $(1 - \beta)$.

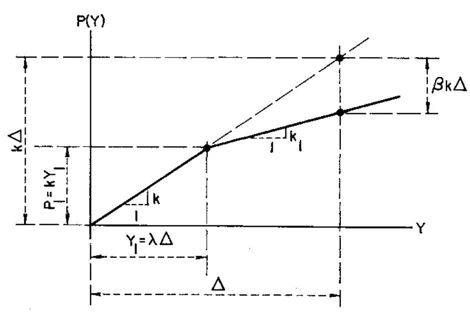


Fig. 5 Bilinear spring

As an illustration, the S^* and $\hat{S}$ spectra are shown in Fig. 4 for the case of a half-sine pulse ground acceleration:

$$\begin{aligned} f(t) &= \sin\pi(t/t_0) \qquad (0 \leq t \leq t_0) \\ &= 0 \qquad (t > t_0) \end{aligned} \tag{33}$$

Variations in the pulse duration are accounted for by using the dimensional frequency parameter $2ft_m = \omega t_m/\pi$ as the abscissa, where $t_m = t_0/2$ is the rise time of pulse. The characteristic time T_c is taken equal to t_m in defining the spectrum $\hat{S}$.

5. Bilinear Spring

As an illustration of another type of nonlinear spring, consider one with bilinear characteristics as shown in Fig. 5. The essential features of this discussion are independent of the effects due to hysteresis recovery of the spring, and so the analysis of that phase of the response will be omitted. The spring force is

$$\begin{aligned} P(Y) &= KY \qquad |Y| \leq Y_1 \quad \text{(34a)} \\ &= KY - (K - K_1)(Y - Y_1) \qquad |Y| \geq Y_1 \quad \text{(34b)} \end{aligned}$$

Choose the dimensionless variables

$$Z = Y\omega^2/\alpha \qquad Z_1 = Y_1\omega^2/\alpha \qquad \tau = \omega t \tag{35}$$

In terms of these variables, the equation of motion is

$$(d^2Z/d\tau^2) + Z = -f(\tau/\omega) \qquad (|Z| \leq Z_1) \quad \text{(36a)}$$

$$(d^2Z/d\tau^2) + Z - [1 - (K_1/K)](Z - Z_1) = -f(\tau/\omega) \quad \text{(36b)}$$
$$(|Z| \geq Z_1)$$

For given values of ω, K_1/K, and Z_1, this equation can be solved to obtain $|Z|_m$.

In this example, it requires two nonlinearity parameters to define the spring characteristics. One possible choice is to take K_1/K as one parameter and

$$\lambda = Y_1/|Y|_m \tag{37}$$

as the other. Since $Y_1/|Y|_m = Z_1/|Z|_m$, the magnitude of λ corresponding to a given solution of Eq. (36) is simply

$$\lambda = Z_1/|Z|_m \tag{38}$$

Recall that $|Z|_m = S^*$. To obtain sufficient data to establish the spectrum curve for S^*, the solution of Eq. (36) is repeated for a range of values of the parameters ω, K_1/K, and λ.

Another possible choice is to take the nonlinearity parameter β instead of K_1/K but to retain λ as the other parameter. From Eqs. (20) and (34) one has

$$\beta = 1 - [P(\Delta)/K\Delta] = [1 - (K_1/K)](1 - \lambda) \tag{39}$$

This shows that one can change from the parameter K_1/K to β very simply. In fact, for a given λ, the spectrum curve can be designated either by its K_1/K value or by its β value.

The absolute acceleration spectrum is

$$\bar{S} = \max|Z - [1 - (K_1/K)](Z - Z_1)| \tag{40}$$

If $|Z|_m$ occurs within the monotonic range of the spring, this reduces to the simple relation

$$\bar{S} = (1 - \beta)\cdot S^* \tag{41}$$

Observe that, for the forementioned choice of parameters, the spectra are independent of α but are functions of the two parameters λ and β (or K_1/K). In the special case of a perfectly elastic-plastic spring for which $K_1 = 0$, one has $\beta = 1 - \lambda$, and hence only one parameter is necessary to describe the behavior.

Spectra for the response due to a half-sine pulse ground acceleration, as defined by Eq. (33), are shown in Figs. 6–8. The data for plotting these curves are taken from Ref. 13.

Figures 6 and 7 give $\bar{S} = |\ddot{X}|_m/\alpha$ for $\lambda = 0.75$ and 0.25, respectively, with β as the family parameter. The corresponding values of K_1/K [see Eq. (39)] also are indicated on the curves.

Instead of plotting spectra for a fixed value of λ as is done in Figs. 6 and 7, it is useful in some applications to plot the curves for a given value of K_1/K with λ as the family parameter. An example of this form is shown in Fig. 8, which is for $K_1/K = \frac{1}{2}$. All three of the nonlinear spectra, S^*, $\bar{S}$, and $\hat{S}$, are given.

6. Use of Spectra in Analysis and Design Problems

a. Determination of Response

The problem of determining the response of a given system when subjected to a ground acceleration of a specified functional form, $f(t)$, is discussed in this section. By a given system is meant one for which the spring force, $P(Y)$, is completely specified as well as the magnitudes of m and ω. It is assumed that the two nonlinear spectra S^* and $\bar{S}$ are given also. If the $\hat{S}$ spectra is available, it can be used in place of the S^* spectra. For definiteness in the discussion it will be assumed that the spectra are based on the use of the nonlinearity parameter β, which is defined by Eq. (24).

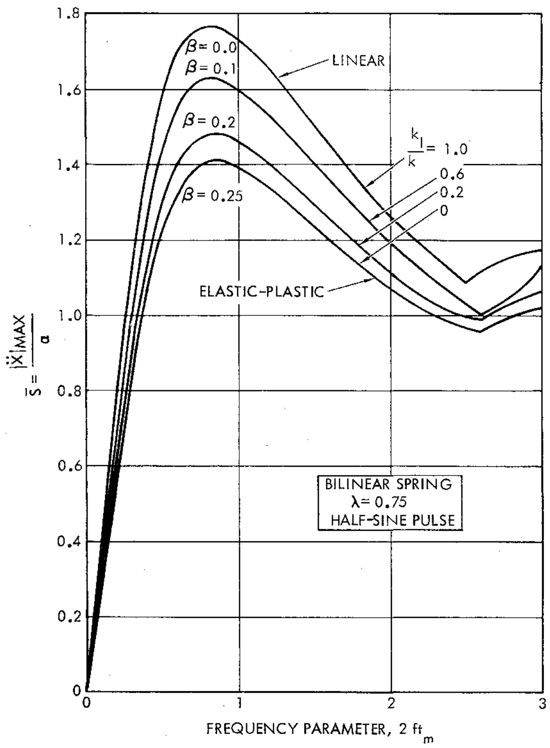


Fig. 6 Shock spectra, $\bar{S}$ (bilinear spring $\lambda = 0.75$ half-sine pulse)

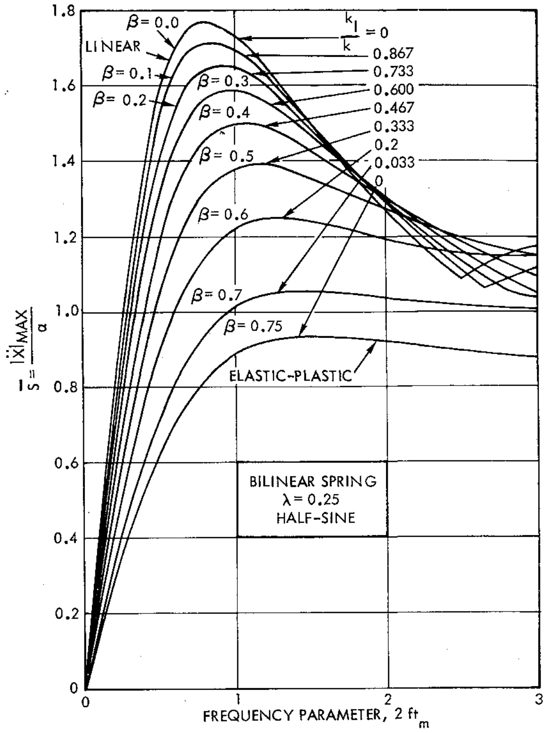


Fig. 7 Shock spectra, $\bar{S}$ (bilinear spring $\lambda = 0.25$ half-sine)

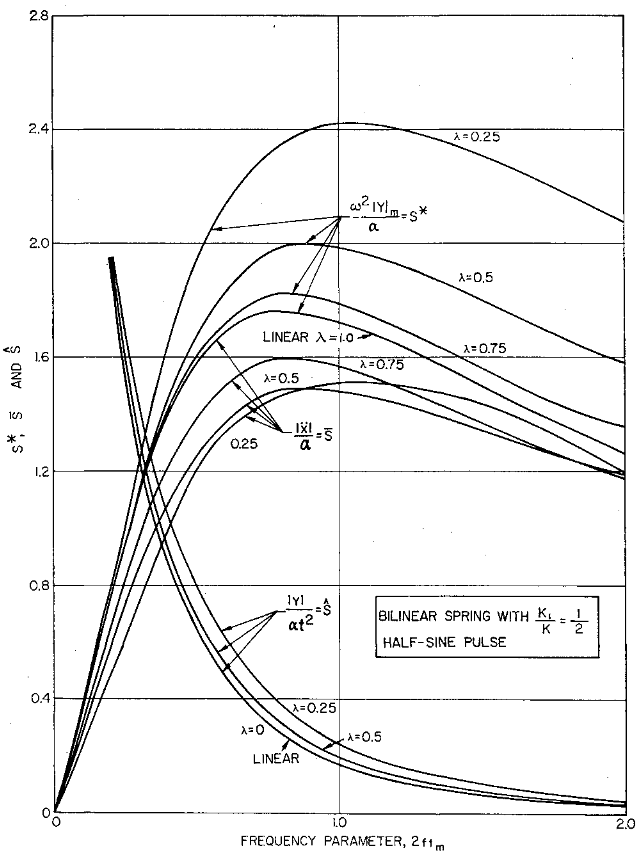


Fig. 8 Shock spectra; bilinear spring

Response problems may be classified into three types, depending upon the data that are given. These are listed below together with an outline of method of solving each.

Type 1

Given $|Y|_m = \Delta$, find α and $|\ddot{X}|_m$.

Procedure: Calculate $\beta = 1 - [P(\Delta)/K\Delta]$. Corresponding to this β and the given ω, read the value of S^* from the spectrum curve. Calculate $\alpha = \omega^2\Delta/S^*$. Read the corresponding value of $|\ddot{X}|_m/\alpha$ from the $\bar{S}$ spectrum. Alternatively, if the response is within the monotonic range of the spring, one may calculate $|\ddot{X}|_m = (1 - \beta)\cdot\alpha S^*$.

Type 2

Given α, find $|Y|_m$ and $|\ddot{X}|_m$.

Procedure: Assume $|Y|_m$. Determine α as for type 1. Repeat for a range of values of $|Y|_m$. Plot α vs $|Y|_m$, and

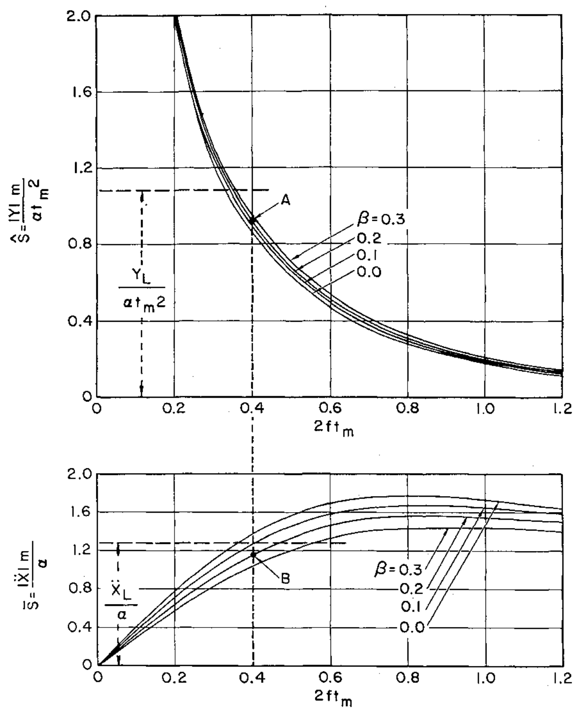


Fig. 9 Schematic application of $\bar{S}$ and $\hat{S}$ spectra to spring design

thus determine the value of $|Y|_m$ corresponding to the given value of α. Determine $|\ddot{X}|_m$ corresponding to this $|Y|_m$ as for type 1.

Type 3

Given $|\ddot{X}|_m$, find α and $|Y|_m$.

Procedure: Assume $|Y|_m$. Determine $|\ddot{X}|_m$ and α as for type 1. Repeat for a range of values of $|Y|_m$. Plot $|Y|_m$ and α vs $|\ddot{X}|_m$, and thus determine their values for the given value of $|\ddot{X}|_m$.

b. Spring Design

Considered here is the problem of designing a spring so that both $|Y|_m$ and $|\ddot{X}|_m$ are equal to or less than specified magnitudes for a given acceleration $\ddot{s}(t) = \alpha f(t)$.

For definiteness, take as an example a cubic-softening spring with the nonlinearity expressed in terms of the parameter β. Assume that the $\bar{S}$ and $\hat{S}$ spectra are available, as shown schematically in Fig. 9, and that the magnitudes of m, β, and t_m are given. Denoting the specified maximum displacement and acceleration responses by Y_L and $\ddot{X}_L$, the design specifications are that $|Y|_m \leqq Y_L$ and $|\ddot{X}|_m \leqq \ddot{X}_L$.

Proceed in the following manner. Draw a horizontal line on the $\hat{S}$ spectra at a distance $Y_L/\alpha t_m^2$ above the origin. Any point on the S curves below this line will fulfill the displacement response specification.

Draw a horizontal line on the $\bar{S}$ spectra at a distance $\ddot{X}_L/\alpha$ above the origin. Any point on the $\bar{S}$ curves below this line will fulfill the acceleration response specification.

Any two points, such as A and B in Fig. 9, at the same frequency and for the same β, will fulfill both requirements. Within these limitations one may select the frequency $\omega = 2\pi f$ and the nonlinearity β to obtain an acceptable design. From the selected values of ω and β, the spring constants K and K_3 can be calculated.

References

[1] Barton, M. V. (ed.), "Shock and structural response, a collection of papers presented at a colloquium on shock and vibration at the Annual Meeting of the ASME, 30 November 1960," Am. Soc. Mech. Engrs. monogram (1960).

[2] Jacobsen, L. S. and Ayre, R. S., *Engineering Vibrations* (McGraw-Hill Book Co. Inc., New York, 1958), Chap. 4.

[3] Frankland, J. M., "Effect of impact on simple elastic structures," Proc. Soc. Exptl. Stress Anal. **6,** 7–27 (1949).

[4] Ayre, R. S., "Transient response to step and pulse functions," *Shock and Vibration Handbook* (McGraw-Hill Book Co. Inc., New York, 1961), Vol. 1, Chap. 8.

[5] Fung, Y. C. and Barton, M. V., "Some characteristics and uses of shock spectra," J. Appl. Mech. **25,** 365–372 (1958).

[6] Mindlin, R. D., "Dynamics of package cushioning," Bell System Tech. J. **24,** 353–461 (July–October 1945).

[7] Thomson, W. T., "Shock spectra of a nonlinear system," Space Technology Labs. Inc., EM 9-1 (January 1959); also J. Appl. Mech. **27,** 528–534 (1960).

[8] Velestos, A. S. and Newmark, N. M., "Effect of inelastic behavior on the response of simple systems to earthquake motions," *Proceedings of the Second World Conference on Earthquake Engineering* (Science Council of Japan, Tokyo, 1960), Vol. 2, pp. 895–912.

[9] Newmark, N. M., "An engineering approach to blast resistance design," Trans. Am. Soc. Civil Engrs. **121** (1956).

[10] Fung, Y. C. and Barton, M. V., "Shock response of a nonlinear system," J. Appl. Mech. **29,** 465–476 (1962).

[11] Fung, Y. C. and Barton, M. V., "The performance of nonlinear systems subjected to ground shock," Aerospace Eng. **21,** 46–57 (February 1962).

[12] Barton, M. V., Chobotov, V., and Fung, Y. C., "A collection of information on shock spectrum of a linear system," Space Technology Labs. Inc. Rept. EM 11-9 (July 1961).

[13] Fung, Y. C. and Schreiner, R. N., "Tables and charts of the shock response of nonlinear systems," Space Technology Labs. Inc., Part I, EM 10-23 (1960); Part II, EM 11-10 (1961); Part III, EM 12-15 (1962).